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Some Formulas in Invariant Theory

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0. INTRODUCTION

In an earlier paper [2] the author discussed the relationship between invariants of the group $G = Z/pZ$ in characteristic p and the classical covariants of $SL(2,k)$ in characteristic zero. This is more closely studied here in the corresponding representation rings. In characteristic zero the only indecomposable (= irreducible) $SL(2,k)$-modules are the R_d: s, where R_d is the set of homogeneous forms of degree d of the polynomial ring $R = k[x,y]$. In characteristic p there are p indecomposable G-modules $V_1, \ldots, V_p$. The corresponding representation rings are

$$R_{SL(2,k)} \cong Z[X] \qquad \text{with} \quad R_1 \to X$$

and

$$R_G \cong Z[X]/_{P(X)} \qquad \text{with} \quad V_2 \to X$$

(here $P(X)$ is a polynomial of degree p). There is a natural ringhomomorphism that maps $R_d \to V_{d+1}$ for $d < p$. Thus, many of the computations made in R_G (see [1, 3]; can be lifted to $R_{SL(2,k)}$ and usually things are much simpler there. Many of the symmetries in R_G do not occur in $R_{SL(2,k)}$.

In Section 1 the classical invariants and covariants in characteristic zero are treated. First, there is a simple proof of a recent formula by Springer [10] for the Hilbert series of the ring of invariants I_d of $SL(2,k)$ of a binary form of degree d. Then the technique of computation in $R_{SL(2,k)}$ is developed. Formulas for the decomposition of symmetric and exterior powers are obtained. The invariants occur as the first component of this decomposition.

An explicit formula for the Hilbert series of the ring of covariants C_d and an estimate of the coefficients is found. It follows that the rings I_d and C_d are Gorenstein. We get a generalization of Molien's theorem for the invariants (of a finite group) of the differential forms with polynomial coefficients.

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In Section 2 some of the previous results in [3] are extended to the representation ring R_G (here $G = Z/_{pZ}$ and the characteristic of k is p). Springer's partial fractions [11] are used to find an explicit formula for the Hilbert series of the invariants. The result of the computations is stated for six and seven variables for certain classes of primes p.

1. Classical Invariants and Covariants

Let k be a field of characteristic zero and $G = SL(2, k)$. We refer to Springer [10] for the basic results. The only irreducible $k[G]$-modules are R_d for $d = 1, 2 \ldots$, the homogeneous forms of $R = k[x, y]$ of degree d. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $gx = ax + by$ and $gy = cx + dy$ and $gf(x, y) = f(gx, gy)$ for $f \in R_d$. Let $S^{\cdot} V$ denote the symmetric k-algebra of the vector space V.

Definition 1.1. The classical ring of invariants.

$$I_d = \{ f \in S^{\cdot} R_d ; gf = f \text{ for all } g \in G \}.$$

The first "main problem" of invariant theory is to find the generators and the relations for the rings I_d. This has been done for small d, but in general it is a hopeless task. A more reasonable problem is to find the Hilbert series of the graded ring I_d. In general if $A = \sum_{n \geq 0} A_n$ is a graded k-algebra with $A_0 = k$, we denote the *Hilbert series* of A by

$$F(A, t) = \sum_{n \geq 0} \dim_k A_n t^n.$$

Recently Springer [11] has found an explicit formula (suitable for a computer) for $F(I_d, t)$.

Let $n > 0$ be an integer and consider the field extension

$$k(t^n) \hookrightarrow k(t),$$

where $k(t)$ is the field of rational functions.

Definition 1.2. Let $\Phi_n : k(t) \to k(t^n)$ denote the "Reynolds operator" defined by

$$(\Phi_n(h))(t^n) = \frac{1}{n} \sum_{\gamma \in \mu_n} h(\gamma t)$$

where μ_n is the group of nth roots of unity. We give a simplified proof of the following:

THEOREM 1.3 (Springer [11])

$$F(I_d, t) = \sum_{0 \leqslant i < d/2}' (-1)^i \Phi_{d-2i} \times \left(\frac{t^{i(i+1)}}{(1-t^4)(1-t^6)\cdots(1-t^{2(d-i)})(1-t^2)\cdots(1-t^{2i})} \right).$$

Proof. Let

$$f_i(t) = t^{i(i+1)/2} \frac{(1-t^d)(1-t^{d-1})\cdots(1-t^{d-i+1})}{(1-t)(1-t^2)\cdots(1-t^i)}$$

and

$$F(I_d, t) = \sum_{n \geqslant 0} c_n t^n.$$

Then by formula (7) of [10, p. 64] we have

$$\begin{aligned} c_n &= \sum_{i=0} (-1)^i \left(\frac{f_i(t)}{(1-t^2)(1-t^3)\cdots(1-t^d)} \right)_{(d-2i)n/2} \\ &= \sum_{i=0}^{d/2} (-1)^i \left(\frac{f_i(t^2)}{(1-t^4)\cdots(1-t^{2d})} \right)_{(d-2i)n} \\ &= \sum_{0 \leqslant i < d/2}' (-1)^i \left(\Phi_{d-2i} \left(\frac{f_i(t^2)}{(1-t^4)\cdots(1-t^{2d})} \right) \right)_n \\ &= \sum_{0 \leqslant i < d/2} (-1)^i \left(\Phi_{d-2i} \left(\frac{t^{i(i+1)}}{(1-t^4)\cdots(1-t^{2(d-i)})(1-t^2)\cdots(1-t^{2i})} \right) \right)_n. \end{aligned}$$

Here $(\sum a_j t^j)_n = a_n$. We also used the following:

$$\Phi_m \left(\sum_{j \geqslant 0} a_j t^j \right) = \sum_{j \geqslant 0}' a_{mj} t^j.$$

Q.E.D.

There are many similarities between the invariant theory of $G = SL(2, k)$ in characteristic zero and of Z/pZ in characteristic $p > 0$ (see [2]). This is most conveniently seen in the representation ring R_G, i.e., the free abelian group on $R_0, R_1, R_2, \ldots$ and the multiplication induced by the tensor product over k. Thus, $g \cdot (v \otimes w) = gv \otimes gw$ for $v \in V$ and $w \in W$. To better see the analogy we change the notation and put

$$V_n = R_{n-1}.$$

Hence $\dim_k V_n = n$ and the basic relation is $V_2 \otimes V_n = V_{n+1} \oplus V_{n-1}$ (a special case of the Clebsch–Gordan formula). As in [1] we introduce a new variable μ by

$$V_2 = \mu + \mu^{-1}.$$

Furthermore, we denote by U_n the nth-degree second Chebyshev polynomial defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\,\theta}{\sin\theta}.$$

Then one easily gets the following result (the proof is a copy of the characteristic p proof only simpler because there are no restrictions on n).

THEOREM 1.4. (a) *R_G is generated by V_2. We have $V_{n+1} = U_n(V_2/2)$.*

(b) *$R_G \cong Z[X]$, the isomorphism given by $V_2 \to X$.*

(c) *$V_n = (\mu^n - \mu^{-n})/(\mu - \mu^{-1})$ in $R_G[\mu]$.*

Remark 1.5. The polynomial $U_n(X/2)$ has integer coefficients. We need the *homogeneous Gaussian polynomials*:

DEFINITION 1.5.

$$G_{n,r}(X, Y) = \frac{(X^n - Y^n)\cdots(X^{n-r+1} - Y^{n-r+1})}{(X^r - Y^r)\cdots(X - Y)}.$$

There are two generating functions:

$$\prod_{j=1}^{n} (1 + X^{n-j}Y^{j-1}t) = \sum_{r=0}^{n} (XY)^{(r(r-1)/2)}\, G_{n,r}(X, Y)\, t^r, \qquad (*)$$

$$\prod_{j=0}^{n} (1 - X^{n-j}Y^{j}t)^{-1} = \sum_{r=0}^{\infty} G_{r+n,n}(X, Y)\, t^r. \qquad (**)$$

Let $\wedge^r V$ ($S^r V$) denote the rth exterior (symmetric) power of the vector space V. Let further

$$\lambda_t(V) = \sum_{r \geqslant 0} \wedge^r V\, t^r \qquad \text{and} \qquad \sigma_t(V) = \sum_{r \geqslant 0} S^r V\, t^r.$$

THEOREM 1.6. (a) *$\lambda_{-t}(V_n)\,\sigma_t(V_n) = 1$ in $R_G[[t]]$.*

(b) $\lambda_t(V_{n-1}) = \sum_{j=0}^{n} (1 + \mu^{n-2j}t)$ *in $R_G[\mu][t]$.*

(c) $\sigma_t(V_{n-1}) = \prod_{j=0}^{n} (1 - \mu^{n-2j}t)^{-1}$ *in $R_G[\mu][[t]]$.*

Proof. (a) There is a split exact sequence (for $r > 0$) (see [1, Proposition II.2.3])

$$0 \to \bigwedge^r V \to \bigwedge^{r-1} V \otimes S^1 V \to \cdots \to \bigwedge^1 V \otimes S^{r-1} V \to S^r V \to 0.$$

It follows that the coefficient of t^r of $\lambda_{-t}(V)\, \sigma_t(V)$ is zero.

(b) follows by induction on n as in [1, Proposition III.1.3].

(c) follows immediately from (a) and (b). Q.E.D.

Using (*) and (**) with $X = \mu$ and $Y = \mu^{-1}$ we get

COROLLARY 1.7. (a) $\bigwedge^r V_n = G_{n,r}(\mu, \mu^{-1})$,

(b) $S^r V_{n+1} = G_{n+r,n}(\mu, \mu^{-1})$.

COROLLARY 1.8. (Generalized Hermite Reciprocity Theorem).

$$S^r V_{n+1} \cong S^n V_{r+1}.$$

Remark 1.9. *Let* $S^r V_{n+1} = c_1 V_1 + c_2 V_2 + \cdots$ be the decomposition of $S^r V_{n+1}$. Then we get $c_1 = \dim_k(S^r V_{n+1})^G = \dim_k(I_n)_r = \dim_k(I_r)_n$ which is the classical Hermite Theorem.

RECIPROCITY THEOREM 1.10. $\sigma_{1/t}(V_n) = (-t)^n \sigma_t(V_n)$.

Remark 1.11. In characteristic $p > 0$ the analogous reciprocity theorem (see [3, Theorem 2.6]) implies new symmetry relations among the $S^r V_{n+1}$: s, but here it seems as there are no such consequences.

DEFINITION 1.12. $\sigma_t(V_{n+1}) = \sum_{j=1}^{\infty} f_{n,j}(t)\, V_j$.

Here the $f_{n,j}(t)$: s are rational functions with non-negative integer coefficients. They were introduced in [1, Chap.V.2] without the authors realizing that they could be useful in characteristic zero. An example is $f_{3,2} = 0$ (see [1, Lemma V.2.5]). This means that $S^r V_4$ never has a V_2 in its decomposition. The most interesting of the $f_{n,j}$: s is $f_{n,1}(t) = F(I_n, t)$ which is the Hilbert series of the ring of invariants. In [1, V.4.3] the following is proved:

PROPOSITION 1.13.

$$f_{n,v}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \prod_{j=0}^{n} (1 - e^{(n-2j)i\varphi} t)^{-1} \sin v\varphi \sin \varphi \, d\varphi.$$

COROLLARY 1.14. (Molien–Weyl Formula for $SL(2, k)$).

$$F(I_d, t) = \frac{1}{\pi}\int_{-\pi}^{\pi} \prod_{j=0}^{d} (1 - e^{(d-2j)i\varphi} t)^{-1} \sin^2 \varphi \, d\varphi.$$

THEOREM 1.15.

$$f_{n,v}(t) = \sum_{0 \leqslant j < n/2} (-1)^j \Phi_{n-2j} \left(\frac{t^{v-1+j(j+1)}}{(1-t^4)\cdots(1-t^{2(n-j)})(1-t^2)\cdots(1-t^{2j})} \right).$$

Proof. We use the following decomposition into partial fractions (see Springer [11, Proposition 1]):

$$\prod_{j=0}^{n} (1 - \mu^{n-2j} t)^{-1}$$
$$= \sum_{0 \leqslant j < n/2} (-1)^j \Phi_{n-2j} \left\{ \frac{t^{j(j+1)}}{(1-t^2)\cdots(1-t^{2(n-j)})(1-t^2)\cdots(1-t^{2j})} \left(\frac{1}{1-\mu t} - \frac{1}{1-\mu t^{-1}} \right) \right\}.$$

We want to find the $f_{n,v}$ in

$$(\mu - \mu^{-1}) \prod_{j=0}^{n} (1 - \mu^{n-2j} t)^{-1} = \sum_{v=1}^{\infty} f_{n,v}(t)(\mu^v - \mu^{-v})$$

(since $V_v = (\mu^v - \mu^{-v})/(\mu - \mu^{-1})$) so we compute

$$(\mu - \mu^{-1}) \left(\frac{1}{1-\mu t} - \frac{1}{1-\mu t^{-1}} \right) = (1 - t^2) \sum_{v=1}^{\infty} t^{v-1}(\mu^v - \mu^{-v}).$$

Putting the two formulas together we get the desired result (Φ_{n-2j} acts only on t). Q.E.D.

Remark 1.16. Putting $v = 1$ we get $f_{n,1}(t) = F(I_n, t)$. This is essentially Springer's proof of Theorem 1.3.

Dually we can make the following:

DEFINITION 1.17. $\lambda_t(V_{n+1}) = \sum_{j \geqslant 1} h_{n,j}(t) V_j$, where $h_{n,j}$ are polynomials with non-negative integer coefficients.

Of particular interest is

$$h_{n,1}(t) = F((\textstyle\bigwedge^{\cdot} V_{n+1})^G, t),$$

the Hilbert series of the invariant differential forms of V_{n+1}. We have not been able to find an explicit formula like Theorem 1.15 for the $h_{n,j}$: s. We have, however, some other results:

PROPOSITION 1.18.

$$h_{n,v}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \prod_{j=0}^{n} (1 + e^{(n-2j)i\varphi} t) \sin v\varphi \sin \varphi \, d\varphi.$$

Proof. Put $\mu = e^{i\varphi}$ in the formal identity

$$(\mu - \mu^{-1}) \prod_{j=0}^{n} (1 + \mu^{n-2j} t) = \sum_{v=1}^{\infty} h_{n,v}(t)(\mu^{v} - \mu^{-v})$$

and then take the vth Fourier coefficient. Q.E.D.

COROLLARY 1.19.

$$F((\wedge^{\cdot} V_{n-1})^G, t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \prod_{j=0} (1 + e^{(n-2j)i\varphi} t) \sin^2 \varphi \, d\varphi.$$

In order to state the next result we need some combinatorics.

DEFINITION 1.20. $A(m, n, r)$ = the number of partitions of m into at most n parts all of size $\leqslant r$.

Put $V(m, n, r) = A(m, n, r) - A(m-1, n, r)$. We have

$$G_{n+r,n}(X, Y) = \sum_{m=0}^{rn} A(m, r, n) X^{rn-m} Y^{m}.$$

PROPOSITION 1.21.

$$h_{n-1,1}(t) = F((\wedge^{\cdot} V_n)^G, t) = \sum_{r \geqslant 0} V\left(\frac{r(n-r)}{2}, n-r, r\right) t^r.$$

More generally

$$h_{n-1,j}(t) = \sum_{r \geqslant 0} V\left(\frac{r(n-r)-j+1}{2}, n-r, r\right) t^r.$$

Proof. By Corollary 1.7 we have

$$\lambda_t(V_n) = \sum_{r=0}^{n} G_{n,r}(\mu, \mu^{-1}) t^r = \sum_{r=0}^{n} \sum_{m=0}^{r(n-r)} A(m, n-r, r) \mu^{r(n-r)-2m} t^r$$

and hence

$$
\begin{aligned}
(\mu-\mu^{-1})\,\lambda_t(V_n) &= \sum_{r=0}^{m}\sum_{m=0}^{r(n-r)} A(m,n-r,r)(\mu^{r(n-r)-2m+1}-\mu^{r(n-r)-2m-1})\,t^r\\
&= \sum_{j\leqslant 1}\left\{\sum_{r\geqslant 0}\left(A\left(\frac{r(n-r)-j+1}{2},n-r,r\right)\right.\right.\\
&\qquad \left.\left.-A\left(\frac{r(n-r)-j-1}{2},n-r,r\right)\right)t^r\right\}(\mu^j-\mu^{-j})\\
&= \sum_{j\geqslant 1}\left(\sum_{r\geqslant 0} V\left(\frac{r(n-r)-j+1}{2},n-r,r\right)t^r\right)(\mu^j-\mu^{-j}).
\end{aligned}
$$

Q.E.D.

PROPOSITION 1.22.

$$
F((S^\cdot V_{n+1}\otimes \textstyle\bigwedge^\cdot V_{n+1})^G,t,s)=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\prod_{j=0}^{n}\left(\frac{1+e^{(n-2j)i\varphi}s}{1-e^{(n-2j)i\varphi}t}\right)\sin^2\varphi\,d\varphi,
$$

where the double grading means that the coefficient of $t^r s^v$ is the dimension of the invariant differential forms of degree v with homogeneous polynomial coefficients of degree r.

Proof. Using Theorem 1.6 we get in $R_G[\mu][s,t]]$

$$
\sigma_t(V_{n+1})\,\lambda_s(V_{n+1})=\prod_{j=0}^{n}\frac{1+\mu^{n-2j}s}{1-\mu^{n-2j}t}=\sum_{v=1}^{\infty} g_{m,v}(s,t)\,\frac{\mu^v-\mu^{-v}}{\mu-\mu^{-1}}.
$$

We want $g_{n,1}(s,t)$ and this we get by putting $\mu=e^{i\varphi}$ and taking the first Fourier coefficient (after multiplication by $\mu-\mu^{-1}$). Q.E.D.

Next we turn to the classical covariants. For the basic definitions and results see Springer [10] and also [2]. Let again $R=k[x,y]$ and R_d the forms of degree d in R.

DEFINITION 1.23. $C_d=(S^\cdot R_d\otimes R)^G$ is the ring of *covariants of a binary form of degree d.*

We compute the Hilbert series, where the grading refers to the first factor.

THEOREM 1.24.

$$
\begin{aligned}
&F(C_d,t)\\
&=\sum_{0\leqslant j<d/2}(-1)^j\Phi_{d-2j}\left(\frac{t^{j(j+1)}}{(1-t^{\varepsilon(d)})(1-t^4)\cdots(1-t^{2(d-j)})(1-t^2)\cdots(1-t^{2j})}\right),
\end{aligned}
$$

where

$$\begin{aligned} \varepsilon(d) &= 1 \qquad \text{if } d \text{ is odd} \\ &= 2 \qquad \text{if } d \text{ is even.} \end{aligned}$$

Proof. Using the exercises on p. 53 in [10] we find that if $F(C_d, t) = \sum_{n=0}^{\infty} c_n t^n$, then

$$c_n = \sum_{j=0}^{d/2} (-1)^j \left(\frac{t^{j(j+1)/2}}{(1-t)\cdots(1-t^{d-j})(1-t)\cdots(1-t^j)}\right)_{[dn/2]-jn}$$

If d is *even*, then the proof runs exactly like the proof of Theorem 1.3.

Let now d be *odd*.

(a) Assume that $n = 2v$. Then

$$\begin{aligned} c_{2v} &= \sum_{j=0}^{d/2} (-1)^j \left(\frac{t^{j(j+1)/2}}{(1-t)\cdots(1-t^{d-j})(1-t)\cdots(1-t^j)}\right)_{v(d-2j)} \\ &= \sum_{0 \leqslant j < d/2} (-1)^j \left(\Phi_{d-2j}\left(\frac{t^{j(j+1)}}{(1-t^2)\cdots(1-t^{2(d-j)})(1-t^2)\cdots(1-t^{2j})}\right)\right)_{2v}. \end{aligned}$$

(b) Assume that $n = 2v+1$. We get $[dn/2] - jn = (d-2j)v + (d-1)/2 - j$ and hence

$$\begin{aligned} c_{2v-1} &= \sum_{0 \leqslant j < d/2} (-1)^j \left(\frac{t^{j(j+1)/2+j-(d-1)/2}}{(1-t)\cdots(1-t^{d-j})(1-t)\cdots(1-t^j)}\right)_{(d-2j)v} \\ &= \sum_{0 \leqslant j < d/2} (-1)^j \left(\Phi_{d-2j}\left(\frac{t^{j(j+1)/2+j-(d-1)/2}}{(1-t)\cdots(1-t^{d-j})(1-t)\cdots(1-t^j)}\right)\right)_v \\ &= \sum_{0 \leqslant j < d/2} (-1)^j \left(\Phi_{d-2j}\left(\frac{t^{j(j+1)+2j-d+1+d-2j}}{(1-t^2)\cdots(1-t^{2(d-j)})(1-t^2)\cdots(1-t^{2j})}\right)\right)_{2v+1} \end{aligned}$$

Thus, we get

$$\sum_{n \text{ even}} c_n t^n = \sum_{0 \leqslant j < d/2} (-1)^j \Phi_{d-2j}\left(\frac{t^{j(j+1)}}{(1-t^2)\cdots(1-t^{2(d-j)})(1-t^2)\cdots(1-t^{2j})}\right)$$

and

$$\sum_{n \text{ odd}} c_n t^n = \sum_{0 \leqslant j < d/2} (-1)^j \Phi_{d-2j}\left(\frac{t^{j(j+1)+1}}{(1-t^2)\cdots(1-t^{2(d-j)})(1-t^2)\cdots(1-t^{2j})}\right).$$

Addition gives the desired formula. Q.E.D.

Fix $d > 0$. We want to estimate the growth of c_n in $F(C_d, t) = \sum_{n \geq 0} c_n t^n$. The computations are close to the proof of Propositions 3.4.8 and 3.4.9 in [10].

PROPOSITION 1.25. *Let* $F(C_d, t) = \sum_{n \geq 0} c_n t^n$. *Then* $c_n = An^{d-1} + O(n^{d-2})$, *where*

$$A = \frac{2}{\pi} \cdot \frac{1}{d!} \int_0^\infty \left(\frac{\sin x}{x}\right)^d dx.$$

Proof. First we assume that d is *even*. By the proof of Theorem 1.24 we have

$$\begin{aligned} c_n &= \sum_{j=0}^{d/2} (-1)^j \left(\frac{t^{j(j+1)/2}}{(1-t)\cdots(1-t^{d-j})(1-t)\cdots(1-t^j)}\right)_{dn/2-jn} \\ &= \sum_{j=0}^{d/2} (-1)^j \left(\frac{f_j(t)}{(t-t)\cdots(1-t^d)}\right)_{dn/2-jn}, \end{aligned}$$

where

$$f_j(t) = t^{j(j+1)/2} \frac{(1-t)^d \cdots (1-t^{d-j+1})}{(1-t)\cdots(1-t^j)}.$$

Using Lemma 1.26 below and that $f_j(1) = \binom{d}{j}$ we get

$$c_n = An^{d-1} + O(n^{d-2}),$$

where

$$A = \frac{1}{d!(d-1)!} \sum_{j=0}^{d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1}.$$

Lemma 1.27 below finishes the proof.

The case when d is *odd* is very similar only $dn/2 - jn$ is replaced by $(d/2 - j)(n-1)$ and $f_j(t)$ by $f_j(t) \cdot t^{j-(d-1)/2}$ which does not change anything; we get the same A. Q.E.D.

LEMMA 1.26. *Consider the expansion* (*where* $g(1) \neq 0$)

$$\frac{g(t)}{\prod_{i=1}^d (1-t^{m_i})} = \sum_{n \geq 0} c_n t^n.$$

Then

$$c_n = An^{d-1} + O(n^{d-2})$$

where

$$A = \frac{g(1)}{(\prod_{i=1}^{d} m_i)(d-1)!}.$$

Proof. Put $1 - t = s$ and write down the Laurent series in s around zero. Then use

$$s^{-d} = (1-t)^{-d} = \sum_{n \geqslant 0} \binom{n+d-1}{d-1} t^n$$

and

$$\binom{n+d-1}{d-1} = \frac{n^{d-1}}{(d-1)!} + O(n^{d-2}).$$

LEMMA 1.27.

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^d dx = \frac{\pi}{2(d-1)!} \sum_{j=0}^{d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1}.$$

Proof.

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^d dx = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\sin z}{z}\right)^d dz = \frac{1}{2} \int_L \left(\frac{\sin z}{z}\right)^d dz,$$

where L is a path outside of the origin parallel with the real axis (this idea was told to me by Per-Olof Brundell some 25 years ago during a chess game). We expand

$$\left(\frac{\sin z}{z}\right)^d = \frac{1}{(2i)^d} \cdot \frac{1}{z^d} \sum_{j=0}^{d/2} (-1)^j \binom{d}{j} e^{(d-2j)iz}.$$

Thus, we are left with computing

$$\int_L \frac{e^{iaz}}{z^d} dz = 2\pi i \operatorname*{Res}_{z=0} \frac{e^{iaz}}{z^d} = \frac{(ia)^{d-1}}{(d-1)!}.$$

We close the path with a large semicircle in the upper half plane if $a > 0$ and choose L such that the origin is inside the contour. We get

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^d dx = \frac{1}{2} \cdot \frac{2\pi i \cdot i^{d-1}}{(d-1)!(2i)^d} \sum_{j=0}^{d/2} (-1)^j \binom{d}{j} (d-2j)^{d-1}.$$

LEMMA 1.28.

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^d dx \sim \frac{1}{2}\sqrt{\frac{6\pi}{d}} \qquad \textit{when } d \textit{ is large.}$$

Proof. Just as in the proof of [10, Lemma 3.4.1] we get the following estimates:

$$\sqrt{d}\int_{\pi/2}^\infty \left(\frac{\sin x}{d}\right)^d dx \leqslant \frac{\sqrt{d}}{d-1}\left(\frac{2}{\pi}\right)^{d-1} \to 0 \qquad \text{as} \quad d \to \infty.$$

With $\varepsilon = (\log d)/\sqrt{d}$ we get

$$\sqrt{d}\int_\varepsilon^{\pi/2} \left(\frac{\sin x}{x}\right)^d dx \leqslant \left(\frac{\sin \varepsilon}{\varepsilon}\right)^d \frac{\pi}{2}\cdot\sqrt{d} \leqslant \frac{\pi\sqrt{d}}{2} e^{-(\log d)^2/6} \to 0$$

as $d \to \infty$. Finally

$$\sqrt{d}\int_0^\varepsilon \left(\frac{\sin x}{x}\right)^d dx = e^{O(d\varepsilon^4)}\int_0^{\varepsilon\sqrt{d}} e^{-dx^2/6}\, dx \to \int_0^\infty e^{-x^2/6}\, dx = \frac{1}{2}\sqrt{6\pi}.$$

Q.E.D.

PROPOSITION 1.29. $\dim C_d = d$.

Proof. This follows from Proposition 1.25.

THEOREM 1.30 (Hochster and Roberts [9]). *The rings I_d and C_d are Gorenstein.*

Proof. The group $G = SL(2, k)$ is linearly reductive and it acts rationally on the polynomial ring $S^\cdot R_d$ ($S^\cdot R_d \otimes R$, respectively). By the theorem of Hochster and Roberts [9] the rings of invariants I_d and C_d are Cohen–Macaulay. Furthermore, both rings are integral domains and

$$F(I_d, t^{-1}) = (-1)^{d-2} t^{d+1} F(I_d, t)$$

(see Springer [11]) and

$$F(C_d, t^{-1}) = (-1)^d t^{d+1} F(C_d, t)$$

(see [2, Theorem 3.1]). Then we can use Stanley's theorem [12, Theorem 4.4] which proves that I_d and C_d are Gorenstein. Q.E.D.

The number of generators of I_d is estimated in [10, Proposition 3.4.9]. We want to find the corresponding result for C_d. Since C_d is a Cohen–Macaulay ring of dimension d, there exist algebraically independent elements $\theta_1,\ldots,\theta_d$ in C_d such that C_d is a free $k[\theta_1,\ldots,\theta_d]$-module. Let θ_j have degree m_j. Then

$$F(C_d,t)=\frac{g(t)}{\prod_{j=1}^{d}(1-t^{m_j})}.$$

By Lemmas 1.26 and 1.27 we find that the rank of C_d as a $k[\theta_1,\ldots,\theta_d]$-module is

$$g(1)=A(d-1)!\prod_1^d m_i=\frac{2}{\pi d}\prod_1^d m_j\int_0^\infty\left(\frac{\sin x}{x}\right)^d dx.$$

Using the fact that $m_j\geqslant 2$ except for one j and Lemma 1.28 we get that rkC_d is larger than a constant $\cdot\, 2^d/d^{3/2}$ that tends to infinity as $d\to\infty$. Thus, C_d is far from being a polynomial ring.

EXAMPLE 1.31. For $d=1$, 2, 3, 4 the generators and the relations for the rings C_d are well known (see [6–8]). For $d=5$ and 6 things are getting much worse. Thus,

$$F(C_5,t)$$
$$=\frac{1+t^2+3t^3+3t^4+5t^5+4t^6+6t^7+6t^8+4t^9+5t^{10}+3t^{11}+3t^{12}+t^{13}+t^{15}}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)}$$

(see [1, Theorem V.2.9]). We have

$$\int_0^\infty\left(\frac{\sin x}{x}\right)^5 dx=\frac{115\pi}{384}$$

and hence $g(1)=2/5\pi\cdot 1\cdot 2\cdot 4\cdot 6\cdot 8\cdot 115\pi/384=46$ (assuming that the degrees of $\theta_1,\ldots,\theta_5$ are 1, 2, 4, 6, 8, respectively).

This suggests there are $\eta_1,\ldots,\eta_{46}$ such that $C_5=\sum_{j=1}^{46}k(\theta_1,\ldots,\theta_5)\,\eta_j$. Gordan has shown that C_5 can be generated by 23 covariants (the η_j: s must be polynomials in these).

Finite Groups in Characteristic Zero

Let k be a field of characteristic zero and V a vector space of dimension $n<\infty$. Let G be a *finite* subgroup of $GL(V)$. Then G acts on the exterior powers $\bigwedge^r V$ via $g\cdot(v_1\wedge\cdots\wedge v_r)=(gv_1)\wedge\cdots\wedge(gv_r)$.

PROPOSITION 1.32.

$$F((\textstyle\bigwedge^\cdot V)^G,s)=\frac{1}{|G|}\sum_{g\in G}\det(1+sg).$$

Proof. Define the linear map $\theta_i\colon \bigwedge^i V \to \bigwedge^i V$ by $\theta_i = (1/|G|) \sum_{g\in G} \bigwedge^i g$. Then

$$\theta_i^2 = \frac{1}{|G|^2}\left(\sum_{g\in G} \bigwedge^i g\right)\left(\sum_{h\in G} \bigwedge^i h\right) = \frac{1}{|G|^2} \sum_{g,h\in G} \bigwedge^i (gh) = \frac{1}{|G|} \sum_{g\in G} \bigwedge^i g = \theta_i.$$

Hence θ_i is a projection and its image is $(\bigwedge^i V)^G$. But $\operatorname{Tr}\theta_i = \dim \operatorname{Im}\theta_i = \dim(\bigwedge^i V)^G$. It follows that

$$\frac{1}{|G|} \sum_{g\in G} \det(1+sg) = \frac{1}{|G|} \sum_{g\in G} \sum_{i=0}^{n} \operatorname{Tr} \bigwedge^i g s^i = \sum_{i=0}^{n} \operatorname{Tr}\left(\frac{1}{|G|} \sum_{g\in G} \bigwedge^i g\right) s^i$$
$$= \sum_{i=0}^{n} \dim(\bigwedge^i V)^G s^i = F((\bigwedge^{\cdot} V)^G, s). \qquad \text{Q.E.D.}$$

We can extend this result to differential forms with polynomial coefficients (for $s=0$ we get Molien's theorem).

THEOREM 1.33.

$$F((S^{\cdot}V \otimes \bigwedge^{\cdot} V)^G; t, s) = \frac{1}{|G|} \sum_{g\in G} \frac{\det(1+sg)}{\det(1-tg)},$$

where t refers to the grading of $S^{\cdot}V$ and s to $\bigwedge^{\cdot} V$.

Proof. The proof is very similar to that of the previous proposition. We consider the projection

$$\theta_{ij} = \frac{1}{|G|} \sum_{g\in G} S^i g \otimes \bigwedge^i g\colon \qquad S^i V \otimes \bigwedge^j V \to S^i V \otimes \bigwedge^j V.$$

Then $\operatorname{Tr}\theta_{ij} = \dim(S^i V \otimes \bigwedge^j V)^G$. Furthermore

$$\frac{1}{|G|} \sum_{g\in G} \frac{\det(1+sg)}{\det(1-tg)} = \sum_{i,j} t^i s^j \frac{1}{|G|} \sum_{g G} \operatorname{Tr} S^j g \operatorname{Tr} \bigwedge^i g = \sum_{i,j} t^i s^j \operatorname{Tr}\theta_{ij}. \quad \text{Q.E.D.}$$

The knowledge of $F((S^{\cdot}V \otimes \bigwedge^{\cdot} V)^G; t, s)$ tells us about the eigenvalues of the matrices in G.

PROPOSITION 1.34. *Let*

$$|G|\, F(S^{\cdot}V \otimes \bigwedge^{\cdot} V)^G; t, -1 + u(1-t))|_{t=1} = \sum_{v} b_v u^v.$$

Then b_v is the number of elements in G having exactly v eigenvalues $= 1$.

Proof. Let the eigenvalues of $g \in G$ be $\lambda_j(g)$ where $j = 1, 2, \ldots, n$. Then the left-hand side is (with $m_g =$ the multiplicity of the eigenvalue 1 of g)

$$\sum_{g \in G} \frac{\det(1+sg)}{\det(1-tg)} = \sum_{g \in G} \prod_{j=1}^{n} \frac{1+s\lambda_j(g)}{1-t\lambda_j(g)} = \sum_{g \in G} \frac{1+s)^{m_g}}{(1-t)^{m_g}} \prod_{\lambda_j(g) \neq 1} \frac{1+s\lambda_j(g)}{1-t\lambda_j(g)}.$$

Now we put $s = -1 + u(1-t)$ and then $t = 1$

$$\sum_{g \in G} u^{m_g} \prod_{\lambda_j(g) \neq 1} \left. \frac{1-\lambda_j(g) + u(1-t)\lambda_j(g)}{1-t\lambda_j()} \right|_{t=1} = \sum_{g \in G} u^{m_g} = \sum_{v} b_v u^v. \quad \text{Q.E.D.}$$

Remark 1.35. The last result is inspired by Bourbaki [4, Exercise V.5.3]. There it is shown that if G is generated by pseudoreflexions, then

$$F(S^{\cdot}V \otimes \wedge^{\cdot} V; t, s) = \prod_{i-1}^{n} \frac{1+st^{d_i-1}}{1-t^{d_i}},$$

where the d_i: s are the degrees of the generators of the polynomial ring $(S^{\cdot}V)^G$. It follows the theorem of Shephard, Todd, and Solomon:

$$\sum_{v} b_v u^v = \prod_{i=1}^{n} (1 + (d_i - 1)u)$$

2. Invariants of Z/pZ in Characteristic $p > 0$

In this section we extend some results of [3]. Let $G = Z/pZ$ and k a field of characteristic p. Let further $V_1, V_2, \ldots, V_p$ be the indecomposable $k[G]$-modules ($\dim V_n = n$) and R_G the representation ring of G.

Remark 2.1. $(V_p - V_{p-1})^2 = 1$ in R_G. This follows from the multiplication table of R_G

$$\begin{aligned} V_p V_p - 2V_{p-1} V_p + V_{p-1} V_{p-1} &= pV_p - 2(p-1) V_p + (p-2) V_p + V_1 \\ &= V_1 = 1. \end{aligned}$$

Definition 2.2. (a) $\lambda_t(V_n) = \sum_{i \geqslant 0} \wedge^i V_n t^i$ in $R_G[t]$.

(b) $\sigma_t(V_n) = \sum_{t \geqslant 0} S^r V_n t^r$ in $R_G[[t]]$.

In an earlier paper [3] we worked with the ring $\tilde{R}_G = R_G / V_p$. In order to prove a formula in $R_G[[t]]$ it is enough to show it in $\tilde{R}_G[[t]]$ and then check it is valid after taking dimensions over k.

THEOREM 2.3. (a) $\lambda_{-t}(V_m)\,\sigma_t(V_m) = 1$ if m is *odd*.

(b) $\lambda_{-t}(V_m)\,\sigma_t(V_m) = (1 - (V_p - V_{p-1})\,t^p)/(1 - t^p)$ if m is *even*.

Proof. (a) By Theorem 2.4 in [3] we have

$$\widetilde{\lambda_{-t}}(V_m)\,\sigma_t(\widetilde{V_m}) = 1 \qquad \text{in } \tilde{R}_G[[t]].$$

Looking at the coefficient of t^r of $\lambda_{-t}(V_m)\,\sigma_t(V_m)$ we get for all $r > 0$

$$\sum_{i=0}^{r} (-1)^i \Lambda^i V_m \otimes S^{r-i} V_m = 0$$

Taking $\dim_k$ we have

$$\sum_{i=0}^{r} (-1)^i \binom{m}{i} \binom{m+r-i-1}{m-1} = 0$$

since it is the coefficient of t^r of

$$(1-t)^m \cdot \frac{1}{(1-t)^m} = 1.$$

Hence $\lambda_{-t}(V_m)\,\sigma_t(V_m) = 1$ also in $R_G[[t]]$ by the remarks before the theorem.

(b) Let now m be even. Then

$$\widetilde{\lambda_{-t}}(V_m)\,\sigma_t(\widetilde{V_m}) = \frac{1 + V_{p-1}t^p}{1 - t^p}$$

in $\tilde{R}_G[[t]]$. Taking the coefficient of t^r and then $\dim_k$ we get (for $r > 0$) zero on the left-hand side and

$$\begin{array}{ll} p & \text{if } p \mid r, \\ 0 & \text{if } p \nmid r \end{array}$$

on the right-hand side. Hence we get

$$\lambda_{-t}(V_m)\,\sigma_t(V_m)$$
$$= \frac{1 + V_{p-1}t^p}{1 - t^p} - \frac{t^p}{1 - t^p} V_p = \frac{1 - (V_p - V_{p-1})\,t^p}{1 - t^p} \qquad \text{in } R_G[[t]]. \quad \text{Q.E.D.}$$

RECIPROCITY THEOREM 2.4. *In $R_G[[t]]$ we have*

$$\sigma_{1/t}(V_{n+1}) = (-t)^{n+1}\sigma_t(V_{n+1}) \qquad \text{if } n \text{ is even}$$
$$= (-t)^{n+1}(V_p - V_{p-1})\,\sigma_t(V_{n+1}) \qquad \text{if } n \text{ is odd.}$$

Proof. If n *even* we get by Theorem 2.3

$$\sigma_t(V_{n+1}) = \frac{1}{\lambda_{-t}(V_{n+1})} = \sum_{j=0}^{n} (1 - \mu^{n-2j}t)^{-1}$$

in $R_G[\mu][[t]]$ where μ is defined by $V_2 = \mu + \mu^{-1}$ (see [1, III.1.3]). Then we only have to replace t by t^{-1}. Next let n be *odd*. Then

$$\sigma_t(V_{n+1}) = \frac{1 - (V_p - V_{p-1})\, t^p}{1 - t^p} \cdot \frac{1}{\lambda_{-t}(V_{n+1})} = \frac{1 - (V_p - V_{p-1})\, t^p}{1 - t^p} \cdot \prod_{j=0}^{n} (1 - \mu^{n-2j}t)^{-1}.$$

Replace t by t^{-1}:

$$\begin{aligned} \sigma_{1/t}(V_{n+1}) &= (-t)^{n+1} \frac{(V_p - V_{p-1}) - t^p}{1 - (V_p - V_{p-1})\, t^p}\, \sigma_t(V_{n+1}) \\ &= (-t)^{n+1}(V_p - V_{p-1})\, \sigma_t(V_{n-1}), \end{aligned}$$

where we used $(V_p - V_{p-1})^2 = 1$. Q.E.D.

COROLLARY 2.5. *Assume that $n + r < p$. Then we have in R_G*

(a) $S^{p-r-n-1}V_{n+1} - S^r V_{n+1} = p^{-1}(\binom{p-r-1}{n} - \binom{r+n}{n})\, V_p$ *if n is even.*

(b) $S^{p-r-n-1}V_{n+1} + (V_p - V_{p-1})\, S^r V_{n+1} = p^{-1}(\binom{p-r-1}{n} + \binom{r+n}{n})\, V_p$ *if n is odd.*

Proof. (a) The formula is true modulo V_p [3, Theorem 2.6a]. Computing the dimensions on both sides we see that the formula is true in R_G.

(b) By the proof of [3, Theorem 2.6.b] we have in $\tilde{R}_G$

$$\widetilde{S^{p-r-n-1}}V_{n+1} = \widetilde{V_{p-1}S^r}V_{n+1}$$

and again we just have to check dimensions. Q.E.D.

Many cases support the fact that formulas for V_{n-1} are much easier when n is even. It might depend on the fact that the V_m:s with odd m generate a ring.

DEFINITION 2.6. $R_G^{\text{odd}} = Z[V_3, V_5, \ldots, V_p]$.

PROPOSITION 2.7. (a) *R_G^{odd} is a ring.*

(b) *R_G^{odd} is generated by V_3 over Z.*

(c) $R_G^{\text{odd}} \cong Z[x]/P(x)$ *where* $V_3 \to x$ *and* $P(x) = (V_3 - 3)\, V_p$ *is a polynomial of degree* $(p+1)/2$

(d) R_G^{odd} *is closed under* $\bigwedge^r$ *and* S^r.

Proof. (a) follows from the multiplication table.

(b) We have $V_3 V_m = V_{m+2} + V_m + V_{m-2}$.

(c) The basic relation is $V_3 V_p = 3V_p$.

(d) follows from Proposition 1.4 in [3]. Q.E.D.

We now turn to the invariant differentials forms of V_{n+1}.

PROPOSITION 2.8. *If n is even, then*

$$F((\textstyle\bigwedge^{\cdot} V_{n+1})^G, t) = p^{-1} \sum_{\gamma \in \mu_p} \prod_{j=0}^{n} (1 + \gamma^{n-2j} t),$$

where μ_p *is the group of* pth *roots of unity.*

Proof. We have

$$\lambda_t(V_{n+1}) = \prod_{j=0}^{n} (1 + \mu^{n-2j} t)$$

and we know that $\bigwedge^r V_{n+1}$ contains only odd V_j: s as composants. Then we can apply the "trace"

$$Tf(\mu) = p^{-1} \sum_{\gamma \in \mu_p} f(\gamma)$$

just as in the proof of Theorem 4.3 in [3]. Q.E.D.

THEOREM 2.9. *If n is even, then*

$$F((S^{\cdot} V_{n+1} \otimes \textstyle\bigwedge^{\cdot} V_{n+1})^G; t, s) = \frac{1}{p} \sum_{\gamma \in \mu_p} \sum_{j=0}^{n} \frac{1 + \gamma^{n-2j} s}{1 - \gamma^{n-2j} t},$$

where μ_p *is the group of pth roots of unity.*

Proof. We have

$$\sigma_t(V_{n+1}) = \sum_{r \geqslant 0} S^r V_{n+1} t^r = \prod_{j=0}^{n} (1 - \mu^{n-2j} t)^{-1}$$

and

$$\lambda_s(V_{n+1}) = \sum_{v \geqslant 0} \wedge^v V_{n+1} s^v = \prod_{j=0}^{n} (1 + \mu^{n-2j} s).$$

It follows that

$$\sum_{r,v \geqslant 0} S^r V_{n+1} \otimes \wedge^v V_{n+1} t^r s^v = \prod_{j=0}^{n} \frac{1 + \mu^{n-2j} s}{1 - \mu^{n-2j} t}.$$

By Proposition 2.7 (a, d) the decomposition of $S^r V_{n+1} \otimes \wedge^v V_{n+1}$ contains only odd V_j: s. Hence we can use the "trace" $Tf(\mu) = p^{-1} \sum_{\gamma \in \mu_p} f(\gamma)$ to get the invariants just as in the proof of Theorem 4.3 in [3]. Q.E.D.

Using the partial fractions of Springer [11] we can give an explicit formula for the Hilbert series of the invariants. For convenience put

$$u_{n,j}(t) = \frac{t^{j(j+1)}}{(1-t^2) \cdots (1-t^{2(n-j)})(1-t^2) \cdots (1-t^{2j})}.$$

THEOREM 2.10. (a) *If n is even, then*

$$F((S^\cdot V_{n+1})^G, t) = \sum_{0 \leqslant j < n/2}' (-1)^j \Phi_{n-2j} \left(\frac{1+t^p}{1-t^p} u_{n,j}(t) \right).$$

(b) *If n is odd, then*

$$F((S^\cdot V_{n+1})^G, t) = \frac{1}{2} \sum_{0 \leqslant j < n/2} (-1)^j \left\{ \Phi_{n-2j} \left(\frac{(1+t)(1+t^{p-1})}{1-t^p} u_{n,j}(t) \right) \right.$$

$$\left. + \frac{1+t^p}{1-t^p} \Phi_{n-2j} \left(\frac{(1+t)(1-t^{p-1})}{1+t^p} u_{n,j}(t) \right) \right\}.$$

Proof. (a) When n is *even*, we have the formula (see [3, Theorem 4.3])

$$F((S^\cdot V_{n+1})^G, t) = p^{-1} \sum_{\gamma \in \mu_p}' \prod_{j=0}^{n} (1 - \gamma^{n-2j} t)^{-1}$$

$$= p^{-1} \sum_{\gamma \in \mu_p} \sum_{0 \leqslant j < n/2} (-1)^j \Phi_{n-2j} \left\{ \left(\frac{1}{1-\gamma t} - \frac{1}{1-\gamma t^{-1}} \right) u_{n,j}(t) \right\}$$

by Springer [11]. Here Φ_{n-2j} acts only on t (not on γ) so we can exchange the order of summation and move $p^{-1} \sum_{\gamma \in \mu_p}$ inside Φ_{n-2j}. Hence

$$p^{-1} \sum_{\gamma \in \mu_p} \left(\frac{1}{1-\gamma t} - \frac{1}{1-\gamma t^{-1}} \right) = \frac{1+t^p}{1-t^p}$$

and the result follows.

(b) Let now n be *odd*. By Theorem 4.14 in [1] we have

$$
\begin{aligned}
&F((S^{\cdot}V_{n+1})^G, t)\\
&\quad = \frac{1}{2p} \sum_{\gamma\in\mu_p} \left(\frac{1+\gamma}{\prod_{j=0}^{n}(1-\gamma^{n-2j}t)} + \frac{1+t^p}{1-t^p} \cdot \frac{1-\gamma}{\prod_{j=0}^{n}(1+\gamma^{n-2j}t)} \right)\\
&\quad = \frac{1}{2p} \sum_{\gamma\in\mu_p} \sum_{0\leqslant j<n/2} (-1)^j \left\{ \Phi_{n-2j} \left((1+\gamma) \left(\frac{1}{1-\gamma t} - \frac{1}{1-\gamma t^{-1}} \right) u_{n,j}(t) \right) \right.\\
&\quad\quad \left. + \frac{1+t^p}{1-t^p} \Phi_{n-2j} \left((1-\gamma) \left(\frac{1}{1+\gamma t} - \frac{1}{1+\gamma t^{-1}} \right) u_{n,j}(t) \right) \right\},
\end{aligned}
$$

where we used Springer's partial fractions and $u_{n,j}(t)$ is an even function. We compute

$$
p^{-1} \sum_{\gamma\in\mu_p} (1+\gamma) \left(\frac{1}{1-\gamma t} - \frac{1}{1-\gamma t^{-1}} \right) = \frac{(1+t)(1+t^{p-1})}{1-t^p}
$$

and

$$
p^{-1} \sum_{\gamma\in\mu_p} (1-\gamma) \left(\frac{1}{1+\gamma t} - \frac{1}{1+\gamma t^{-1}} \right) = \frac{(1+t)(1-t^{p-1})}{1+t^p}
$$

and the result follows. Q.E.D.

Finally we show that the last theorem is useful by computing the Hilbert series for $n=5$ and 6 for certain classes of p.

EXAMPLE 2.11. (a) Let $p \equiv 11 \pmod{30}$ and assume $p = 5q+1 = 3r+2$. Then

$$
F((S^{\cdot}V_6)^G, t) = \frac{N_1}{D_1} - \frac{N_2}{D_2} + \frac{N_3}{D_3},
$$

where

$$
\begin{aligned}
N_1 = {}& 1 + t + 4t^2 + 5t^3 + 11t^4 + 10t^5 + 15t^6 + 12t^7 + 19t^8 + 11t^9 + 14t^{10}\\
&+ 7t^{11} + 9t^{12} + 3t^{13} + 2t^{14} + t^{15} + t^{2q+1}(3 + 7t + 10t^3 + 16t^3\\
&+ 22t^4 + 26t^5 + 29t^6 + 29t^7 + 19t^{10} + 15t^{11} + 10t^{12} + 6t^{13} + 2t^{14})\\
&+ t^{4q+1}(2 + 4t + 9t^2 + 15t^3 + 18t^4 + 24t^5 + 29t^6 + 31t^7 + 24t^{10}\\
&+ 18t^{11} + 11t^{12} + 7t^{13} + 3t^{14} + t^{15}),
\end{aligned}
$$

$$D_1 = (1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)(1-t^p),$$
$$N_2 = (1+t^{2r+1})(t+3t^2+t^3+5t^4+3t^5+4t^6+2t^7+4t^8+2t^9+2t^{10})$$
$$+ t^{2r+2}(1+t+3t^2+3t^3+5t^4+t^5+5t^6+3t^7+3t^8+t^9+t^{10}),$$
$$D_2 = (1-t)(1-t^2)^2(1-t^4)(1-t^8)(1-t^p),$$
$$N_3 = t^6,$$
$$D_3 = (1-t)(1-t^2)(1-t^4)^2(1-t^6)(1-t^p),$$

(b) Let $p = 3q+1$. Then

$$F((S \cdot V_7)^G, t) = \frac{N_1}{D_1} - \frac{N_2}{D_2} + \frac{N_3}{D_3},$$

where

$$N_1 = (1+t^p)(1+t+4t^2+5t^3+5t^4+5t^5+4t^6+t^7+t^8)$$
$$+ 2t^{q+1}(2+3t+4t^2+5t^3+6t^4+3t^5+3t^6+t^7)$$
$$+ 2t^{2q+1}(1+3t+3t^2+6t^3+5t^4+4t^5+3t^6+2t^7),$$
$$D_1 = (1-t)^2(1-t^2)^2(1-t^4)(1-t^5)(1-t^p),$$
$$N_2 = (1+t^p)(2t+t^2+2t^3+t^4+2t^5) + 2t^{(p+1)/2}(1+t+2t^2+2t^3+t^4+t^5),$$
$$D_2 = (1-t)^3(1-t^2)(1-t^3)(1-t^5)(1-t^p),$$
$$N_3 = (1+t^p)\, t^3,$$
$$D_3 = (1-t)^2(1-t^2)^2(1-t^3)(1-t^4)(1-t^p).$$

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